ON CERTAIN PROPERTIES OF THE METRICAL AND GENERALIZED METRICAL GROUPS IN LINEAR SPACES OF n DIMENSIONS

BY

SERGEI ALEXANDER SCHELKUNOFF

INSTRUCTOR IN MATHEMATICS IN THE
STATE COLLEGE OF WASHINGTON

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY, IN THE FACULTY OF PURE SCIENCE OF COLUMBIA UNIVERSITY

1927

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TO

MISS JEAN KENNEDY

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INTRODUCTION

Incidental study of null curves, i. e., curves of zero length, dates back to the middle of the last century. In 1864 Enneper* discovered their remarkable connection with minimal surfaces. Two years later Weierstrass wrote the equations of null curves in a form free from quadratures. During the following fifty years we find various writers making passing remarks concerning isotropic developables, i. e. developable surfaces with a null curve as the edge of regression. Also the idea of null curves was used in stating some theorems of Differential Geometry. But "null" geometry as a body of theorems concerning intrinsic properties of null manifolds did not exist. The first step in this direction was taken by H. Beck (1912) and C. L. E. Moore (1915) who published their researches on intrinsic geometry of isotropic planes and surfaces; they discovered certain invariants and used them to define "distance" and "angle" of the new geometry. These two papers mark the transition period in the development of a new subject.

In 1924 Professor E. Kasner introduced null geometry as a distinct subject into his seminar on Differential Geometry at Columbia University. He suggested the term "null geometry" to designate geometry of a space whose fundamental metrical form is of rank zero to distinguish it from "ordinary" geometry of a space whose fundamental metrical form is of rank equal to its dimensionality. The fundamental problem in this connection is, of course, to discover invariants that could be defined as "distance" and "angle" in such spaces. In 1925 Professor E. Kasner read three papers dealing with various problems of null geometry before the American Mathematical Society (abst. Bull.

* Zeitschrift für Mathematik und Physik., vol. IX (1864), p. 107.

Amer. Math. Soc., vol. XXXI, pp. 214, 226, 293). Professor E. Kasner's seminar of 1926 was also on null geometry. In 1926 J. Lense of Vienna pubished two papers on "ametric manifolds", the term "ametric" being applied to manifolds whose fundamental metrical forms are of ranks less than their respective dimensionalities; he deals with problems concerning immersion of ametric manifolds into ordinary linear manifolds. Thus, his papers are still concerned more with relations of ametric manifolds to ordinary manifolds rather than with intrinsic properties of ametric manifolds.

It is the latter problems that I am mainly concerned with in this paper. The fundamental problem before us is to lay the foundation of "generalized metrical" geometries of non-ordinary spaces.

We shall define the "rank" of a space as the rank of its fundamental metrical form. Spaces of ranks equal to their dimensionalities are called ordinary. Since there is an essential difference between the generalized fundamental metrical forms of spaces (Chap. IV, Sec. 19) of ranks 0 and other non-ordinary spaces I propose to call the latter spaces "aeolotropic" while retaining the term "null" for the former. The term "aeolotropic space" seems appropriate since it will emphasize the heterogeneous character of its metrical invariants. Correspondingly we shall speak of ordinary, aeolotropic, and null geometries.

The method which led me to a generalization of the ordinary orthogonal group (Chap. IV, Sec. 16) is essentially a study of invariance of null subflats under the group of rotations. As the work progressed I discovered several new and interesting properties of the group of rotations; since these results are important and closely related to though do not constitute a part of null geometry I have thought proper to include them in this paper.

The group of rotations in an ordinary flat (F_n) depends upon n^2 coefficients between which there are $\frac{n(n+1)}{2}$

independent relations; hence, these n^2 coefficients are functions of $\frac{n(n-1)}{2}$ independent parameters. In 1846 A. Cayley actually solved the problem; he chose the elements of an arbitrary skew-symmetrical determinant as a set of independent parameters and expressed all the coefficients in terms of them. However, such a set of parameters does not admit any obvious geometrial interpretation. For the case $n = 3$ Euler expressed the coefficients of the corresponding group in terms of three independent angles of rotation; his formulae have important applications in mechanics. I have extended Euler's transformation formulae to any number of dimensions (Chap. I). For this purpose I obtained recurrent formulae of remarkable compactness.

In Chapter II I deal with an "extended" group of rotations; I have applied this name to the group of orthogonal transformations of planes into planes. Further "extended" groups of rotations can be formed if, in general, k-dimensional subflats are taken as elements of transformations. Similarly the "extended" groups of the generalized orthogonal group can be formed. An immediate application of the extended group of rotations was the proof the following theorem. *Every rotation in F_n can be decomposed into a succession of $E\left(\frac{n}{2}\right)$* simple rotations taking place in $E\left(\frac{n}{2}\right)$ ordinary mutually absolutely orthogonal planes; this set of invariant ordinary planes is in general unique.* (Chap. II, Sec. 11). This theorem is a generalization of a well known fact that any rotation in a flat three-dimensional space is a rotation around an axis, or, it would be better to say, a rotation in a certain plane, i. e. there is an invariant plane under every three-dimensional rotation. The first generalization of this theorem was given by F. N. Cole in his article "On rotations in

* Euler's function $E(x)$ is the integral part of x by definition.

space of four dimensions" where he proved the existence of two invariant planes under every four-dimensional rotation. His method, however, cannot be readily extended to higher spaces.

The above theorem can be stated also as a corollary of the following theorem (Chap. III, Sec. 14): *There exist at least n straight lines invariant under every rotation in* F_n. *If n is even these lines are null; if n is odd, one line is ordinary and the rest are null.* The invariant null lines are not absolute invariants, i. e. individual points on these lines are not invariant. I have used this fact for determining the "extent" of component rotations of the general rotation (Chapt. III, Sec. 14).

Chapter IV is devoted to a discussion of invariance of null subflats under the ordinary orthogonal group, generalization of the latter, properties of generalized orthogonants, and algebraic and differential invariants under the generalized orthogonal group. These invariants are Hermitian and have been previously studied but their importance in null geometry was not observed. In aeolotropic spaces mixed invariants make their appearance; H. Beck and C. L. E. More in their researches on geometry of minimal planes discovered invariants which are special cases of the above.

I wish to take this opportunity to thank Mr. K. A. Toropoff, my teacher of mathematics in Orenburg high school. Under his stimulating influence I began to read original mathematical literature and developed a taste for original research. Needless to add that I owe much to other mathematicians with whom I had the pleasure of being personally acquainted. In particular I wish to express my gratitude to Professor Edward Kasner for the inspiring seminar on Differential Geometry where for the first time my attention was called to the subject of null geometry and for his suggestions in preparation of this paper.

CHAPTER I

1. In this paper I shall often refer to a linear ordinary manifold of points as an n-flat F_n.* To avoid frequent repetitions I shall assume that every literal subscript or superscript takes all the integral values from 1 to n unless otherwise stated. When the same letter occurs twice in a product the ordinary summation convention will be assumed and the sign of summation omitted; this is regardless of whether the letter occurs once as a subscript and a superscript or both times as a subscript or both times as a superscript.†

The transformations of the metrical group in F_n I shall write in the form

$$y_i = a_i^k x_k \tag{1}$$

where the coefficients are subject to the well-know conditions

$$\begin{aligned} a_k^i a_k^j &= 0, \quad \text{when } i \neq j, \\ &= 1, \quad \text{when } i = j. \end{aligned} \tag{2}$$

Also, the following equations can be deduced from equations (2):

$$\begin{aligned} a_i^k a_j^k &= 0, \quad \text{when } i \neq j, \\ &= 1, \quad \text{when } i = j. \end{aligned} \tag{3}$$

Equations (2) represent $\frac{n(n+1)}{2}$ independent conditions between n^2 coefficients a_i^k; hence, there must be $\frac{n(n-1)}{2}$

* An n-dimensional manifold is said to be ordinary if the matrix of the coefficients of its metrical fundamental form is of rank n.

† The subscript "n" will be an exception.

independent parameters in terms of which the coefficients a_i^k can be expressed. In 1846 Cayley obtained rational expressions of these coefficients in terms of independent parameters which, however, have no obvious geometrical interpretation.* For the case $n = 3$ Euler's transformations which involve three independent angles of rotation are well-known in geometry and mechanics.

I shall derive reduction formulae for the coefficients of the metrical group in any n-flat in terms of the angles of successive rotations and then point out some interesting properties which will make the process of computation of these coefficients exceedingly simple.

2. I shall note first that equations (1) transform the set of points

$$X_k(0, \cdots, 0, 1_k, 0, \cdots, 0)$$

into

$$Y_k(a_1^k, a_2^k, \cdots, a_n^k).$$

I shall choose the point $X_n(0, 0, \cdots, 0, 1)$ and transform it into the corresponding point Y_n by making $(n-1)$ successive rotations in the planes $X_{1,n}^1, X_{2,n}^2, \cdots, X_{n-1,n}^{n-1}$ respectively through the angles $(1, n), (2, n), \cdots, (n-1, n)$.†

We obtain the following set of equations:

$$\begin{aligned}
y_1 &= x_1 \cos(1\,n) - x_n \sin(1\,n),\\
x_n^1 &= x_1 \sin(1\,n) + x_n \cos(1\,n),\\
y_2 &= x_2 \cos(2\,n) - x_n^1 \sin(2\,n),\\
x_n^2 &= x_2 \sin(2\,n) + x_n^1 \cos(2\,n),\\
&\cdots\cdots\cdots\cdots\\
y_{n-2} &= x_{n-2} \cos(n-2, n) - x_n^{n-3} \sin(n-2, n),\\
x_n^{n-2} &= x_{n-2} \sin(n-2, n) + x_n^{n-3} \cos(n-2, n),\\
y_{n-1} &= x_{n-1} \cos(n-1, n) - x_n^{n-2} \sin(n-1, n),\\
y_n &= x_{n-1} \sin(n-1, n) + x_n^{n-2} \cos(n-1, n).
\end{aligned}$$

* Cayley, A., Mathematical papers, vol. I, pp. 332–335.

† $X_{i,n}^i$ is the plane defined by the coördinate axes OX_i and OX_n^i where $OX_n^1, OX_n^2, \cdots, OX_n^{n-1}$ are successive positions of OX_n^1. (i, n) designates the angle through which OX_i and OX_n^i are rotated.

Making successive substitutions we obtain

$$
\begin{aligned}
y_1 &= x_1 \cos(1\,n) - x_n \sin(1\,n), \\
y_2 &= -x_1 \sin(1n)\sin(2\,n) + x_2 \cos(2\,n) - x_n \cos(1n)\sin(2\,n) \\
&\qquad\qquad 2 < i < n, \\
y_i &= -x_1 \sin(1\,n)\cos(2n)\cos(3\,n) \cdots \cos(i-1, n)\sin(i, n) \\
&\quad - x_2 \sin(2\,n)\cos(3\,n) \qquad \cdots \cos(i-1, n)\sin(i, n) \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
(4) \qquad &\quad - x_{i-1} \sin(i-1, n)\sin(i, n) + x_i \cos(i, n) \\
&\quad - x_n \cos(1\,n)\cos(2\,n) \qquad \cdots \cos(i-1, n)\sin(i, n), \\
y_n &= x_1 \sin(1\,n)(\cos 2\,n)(\cos 3\,n) \cdots \cos(n-1, n) \\
&\quad + x_2 \sin(2\,n)\cos(3\,n) \qquad \cdots \cos(n-1, n) \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad + x_{n-2} \sin(n-2, n)\cos(n-1, n) + x_{n-1} \sin(n-1, n) \\
&\quad + x_n \cos(1\,n)\cos(2\,n) \qquad \cdots \cos(n-1, n).
\end{aligned}
$$

Equations (4) transform as it was stated point X_n into Y_n; at the same time they transform the hyperplane F_{n-1} perpendicular to OX_n into the hyperplane perpendicular to OY_n. If by the general metrical transformation in F_{n-1} the original points

$$
\begin{aligned}
&X_1(1, 0, \cdots, 0), \cdots, X_k(0, \cdots, 0, 1_k, 0, \cdots, 0), \cdots \\
&\qquad\qquad \cdots, X_{n-1}(0, 0, \cdots, 0, 1)
\end{aligned}
$$

were transformed into

$$
\begin{aligned}
&Z_1(b_1^1, b_2^1, \cdots, b_{n-1}^1, 0), \cdots, Z_k(b_1^k, b_2^k, \cdots, b_{n-1}^k, 0), \cdots \\
&\qquad\qquad \cdots, Z_{n-1}(b_1^{n-1}, b_2^{n-1}, \cdots, b_{n-1}^{n-1}, 0)
\end{aligned}
$$

then equations (4) will transform the latter set of points into

$$
\begin{aligned}
&Y_1(a_1^1, a_2^1, \cdots, a_n^1), \cdots, Y_k(a_1^k, a_2^k, \cdots, a_n^k), \cdots \\
&\qquad\qquad \cdots, Y_{n-1}(a_1^{n-1}, a_2^{n-1}, \cdots, a_n^{n-1}).
\end{aligned}
$$

The above coördinates b_k^i are evidently the coefficients of the metrical group in the $(n-1)$-flat.

Thus, from equations (4) we obtain the following reduction formulae:

$$
\begin{aligned}
a_1^k &= b_1^k \cos(1\,n), \\
a_2^k &= -b_1^k \sin(1\,n)\sin(2\,n) + b_1^k \cos(2\,n), \\
(5)\quad a_n^k &= \quad b^k \sin(1\,n)\cos(2\,n)\cos(3\,n)\cdots\cos(n-1,\,n) \\
&\quad + b_2^k \sin(2\,n)\cos(3\,n) \qquad\qquad \cdots\cos(n-1,\,n) \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad + b_{n-1}^k \sin(n-1,\,n)
\end{aligned}
$$

for $k < n$ and

$$
\begin{aligned}
a_i^k = &-b_1^k \sin(1\,n)\cos(2\,n)\cos(3\,n)\cdots\cos(i-1,\,n)\sin(i\,n) \\
&-b_2^k \sin(2\,n)\cos(3\,n) \qquad\qquad \cdots\cos(i-1,\,n)\sin(i\,n) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&-b_{i-1}^k \sin(i-1,\,n)\sin(i\,n) + b_i^k \cos(i\,n);
\end{aligned}
$$

for $k < n$, $2 < i < n$; and

$$
\begin{aligned}
a_1^n &= -\sin(1\,n), \\
a_2^n &= -\cos(1\,n)\sin(2\,n) \qquad\qquad 2 < i < n, \\
a_i^n &= -\cos(1\,n)\cos(2\,n)\cdots\cos(i-1,\,n)\sin(i\,n), \\
a_n^n &= \quad\cos(1\,n)\cos(2\,n)\cdots\cos(n-1,\,n).
\end{aligned}
$$

3. If D_{1k} is a symbol of the operation consisting of making the angles (12), (13), $\cdots$, $(1, k-1)$ equal to zero and then taking the partial derivative with respect to the angle $(1\,k)$, then

$$
(6)\qquad a_i^k = D_{1k}\, a_i^1 \qquad k = 2, \cdots, n.
$$

To prove the theorem apply the operator D_{1k} to equations (5) (noting that if $k = n$, $b_1^1 = 1$, $b_r^1 = 0$ when $r \neq 1$ after the angles ((12), (13), $\cdots$, $(1, n-1)$ are made equal to zero). It is seen then that if the theorem is true for the flat F_{n-1} it is true for F_n. But it is obviously true for F_2. The truth of the theorem can be also seen directly from geometric considerations if we form partial variations of

a_i^1 by giving successively infinitesimal increments to the angles (12), (13), $\cdots$, $(1\,n)$. In this way I find that only the following set of reduction formulae is needed:

$$
\begin{aligned}
a_1^1 &= \quad b_1^1 \cos(1\,n), \\
a_2^1 &= -\,b_1^1 \sin(1\,n)\sin(2\,n) + b_2^1 \cos(2\,n) \\
&\qquad\qquad \text{for } 2 < i < n. \\
a_i^1 &= -\,b_1^1 \sin(1\,n)\cos(2\,n)\cos(3\,n)\cdots\cos(i-1,n)\sin(i\,n) \\
&\quad -\,b_2^1 \sin(2\,n)\cos(3\,n) \qquad\qquad \cdots\cos(i-1,n)\sin(i\,n) \\
(7)\quad &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad -\,b_{i-1}^1 \sin(i-1,n)\sin(i\,n) + b_i^1 \cos(i\,n), \\
a_n^1 &= \quad b_1^1 \sin(1\,n)\cos(2\,n)\cos(3\,n)\cdots\cos(n-1,n) \\
&\quad +\,b_2^1 \sin(2\,n)\cos(3\,n) \qquad\qquad \cdots\cos(n-1,n) \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad +\,b_{n-1}^1 \sin(n-1,n).
\end{aligned}
$$

The labor of computing is still further reduced if we note that

$$
(8) \qquad a_i^1 = D_{in}^{*}\, a_n^1
$$

where D_{in}^{*} is a symbol of the operation consisting of making the angles $(n-1,n)$, $(n-2,n)\cdots(i+1,n)$ equal to zero and taking the partial derivative with respect to the angle (i,n). The truth of the statement is obvious. Thus, the formulae (5) are finally reduced to the following operational equations:

$$
\begin{aligned}
a_n^1 &= \{\sin(1\,n)\cos(2\,n)\cos(3\,n)\cdots\cos(n-1,n)\,D_{1,n-1}^{*} \\
&\qquad + \sin(2\,n)\cos(3\,n)\cdots\cos(n-1,n)\,D_{2,n-1}^{*} \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\qquad + \sin(n-2,n)\cos(n-1,n)\,D_{n-2,n-1}^{*} \\
&\qquad + \sin(n-1,n)\}\, b_{n-1}^1, \\
(9)\quad a_1^i &= D_{in}^{*}\, a_n^1, \qquad\qquad i = 1, 2, \cdots, n-1, \\
a_i^k &= D_{1k}\, a_i^1, \qquad\qquad k = 2, \cdots, n.
\end{aligned}
$$

The tables for the cases $n = 2, 3$, and 4 are given in the appendix. It is possible to obtain symmetrical transformations depending upon angles as independent parameters. Every simply orthogonal set of planes passing through a point in F_{2n} can be decomposed into $(2n - 1)$ absolutely orthogonal sets of planes, n planes in each set.* Performing successive rotations in the planes of these absolutely orthogonal sets we shall obtain the most general rotation. In F_{2n+1} a simply orthogonal set can be decomposed into $(2n + 1)$ absolutely orthogonal sets, n planes in each set and the most general rotation can be obtained as was indicated above.

* Sections 8 and 10.

CHAPTER II

4. Let the matrix of the direction cosines of a set of r mutually orthogonal straight lines passing through a point in $F_n\,(r < n)$ be

$$(10)\qquad \left\| \begin{matrix} a_1^1 & a_1^2 & a_1^3 & \cdots & a_1^n \\ a_2^1 & a_2^2 & a_2^3 & \cdots & a_2^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_r^1 & a_r^2 & a_r^3 & \cdots & a_r^n \end{matrix} \right\|.$$

Then $\dfrac{n!}{r!(n-r)!}$ rth order determinants $a_{1,\cdots,r}^{i_1,\cdots,i_r}$* of this matrix are uniquely determined. On the other hand the quantities $a_{1,\cdots,r}^{i_1,\cdots,i_r}$ determine an infinite number of matrices of the type (10) all of which, however, define sets of r mutually orthogonal lines in the same F_r that is defined by the set of lines (10). Thus, the quantities $a_{1,\cdots,r}^{i_1,\cdots,i_r}$ can be considered as defining the "direction" of F_r in F_n. Here I am especially interested in the case $r = 2$ to a more detailed discussion of which I pass.

5. If the matrix

$$(11)\qquad \left\| \begin{matrix} a_1^1 & a_1^2 & \cdots & a_1^n \\ a_2^1 & a_2^2 & \cdots & a_2^n \end{matrix} \right\|$$

is that of direction cosines of two orthogonal straight lines passing through a point in F_n then the matrix

$$(12)\qquad \left\| \begin{matrix} l_1^1\, a_1^1 + l_1^2\, a_2^1, \cdots, l_1^1\, a_1^n + l_1^2\, a_2^n \\ l_2^1\, a_1^1 + l_2^2\, a_2^1, \cdots, l_2^1\, a_1^n + l_2^2\, a_2^n \end{matrix} \right\|$$

---

* $i_1, \cdots, i_r$ take the values $1, 2, \cdots, n$ subject to the condition $i_1 < i_2 < \cdots < i_r$.

is that of two orthogonal straight lines in the plane determined by the lines (11) provided

$$(l_1^1)^2 + (l_1^2)^2 = 1,$$
$$(l_2^1)^2 + (l_2^2)^2 = 1,$$
$$l_1^1 l_2^1 + l_1^2 l_2^2 = 0.$$

Also, all the second order determinants of the two matrices are correspondingly equal. This is readily verified.

6. A plane in F_n is determined completely by $2(n-2)$ parameters; hence, there must exist

$$\frac{n(n-1)}{2} - 2(n-2) = \frac{n^2 - 5n + 8}{2} \tag{13}$$
$$= \frac{(n-2)(n-3)}{2} + 1$$

invariant relations between its direction cosines.

One relation is

$$a_{1,2}^{i,j}\, a_{1,2}^{i,j} = 1; \qquad i < j \tag{14}$$

and can be immediately deduced if we consider the two rows of the matrix (11) as the first two rows of an n-dimensional orthogonant a_k^i and apply to the latter Laplace's expansion*

$$|a_k^i| = a_{1,2}^{i,j}\, A_{i,j}^{1,2} = 1, \tag{15}$$

where $A_{i,j}^{1,2}$ is the cofactor of $a_{1,2}^{i,j}$. But

$$A_{i,j}^{1,2} = a_{1,2}^{i,j},\dagger$$

hence

$$a_{1,2}^{i,j}\, a_{1,2}^{i,j} = 1.$$

The remaining invariant relations can be deduced from the fact that all the fourth order determinants of the matrix

* Kowalewski, Einführung in die Determinantentheorie, p. 38.

† Kowalewski, p. 163.

$$(16)\qquad \begin{Vmatrix} a_1^1 & a_1^2 & \cdots & a_1^n \\ a_2^1 & a_2^2 & \cdots & a_2^n \\ a_1^1 & a_1^2 & \cdots & a_1^n \\ a_2^1 & a_2^2 & \cdots & a_2^n \end{Vmatrix}$$

vanish.

Using Laplace's expansion of each such determinant along the first two rows we obtain immediately the sought relations:

$$(17)\quad a_{12}^{i_1 i_2}\, a_{12}^{i_3 i_4} - a_{12}^{i_1 i_3}\, a_{12}^{i_2 i_4} + a_{12}^{i_1 i_4}\, a_{12}^{i_2 i_3} = 0 \; (i_1 < i_2 < i_3 < i_4).$$

Not all equations (17) are independent, however.

In the case $n = 4$ there is one relation of the form (17), namely

$$(18)\qquad a_{12}^{12}\, a_{12}^{34} - a_{12}^{13}\, a_{12}^{24} + a_{12}^{14}\, a_{12}^{23} = 0.$$

In the case $n = 5$ we have the following five equations:

$$(19)\qquad \begin{aligned} a_{12}^{12}\, a_{12}^{34} - a_{12}^{13}\, a_{12}^{24} - a_{12}^{14}\, a_{12}^{23} &= 0, \\ a_{12}^{12}\, a_{12}^{35} - a_{12}^{13}\, a_{12}^{25} - a_{12}^{15}\, a_{12}^{23} &= 0, \\ a_{12}^{12}\, a_{12}^{45} - a_{12}^{14}\, a_{12}^{25} - a_{12}^{15}\, a_{12}^{24} &= 0, \\ a_{12}^{13}\, a_{12}^{45} - a_{12}^{14}\, a_{12}^{35} - a_{12}^{15}\, a_{12}^{34} &= 0, \\ a_{12}^{23}\, a_{12}^{45} - a_{12}^{24}\, a_{12}^{35} - a_{12}^{25}\, a_{12}^{34} &= 0, \end{aligned}$$

out of which only three are independent. For instance the fourth equation follows from the first three if we multiply them respectively by a_{12}^{15}, $-a_{12}^{14}$ and a_{12}^{13} and then add.

Thus, in the case $n = 5$ we have four independent relations between 10 direction cosines leaving us 6 independent parameters of a plane in F_5.

7. The conditions which are satisfied by the direction cosines of a plane and a line in it I shall obtain from the fact that all the third order determinants of the matrix

$$\begin{Vmatrix} a_1^1 & a_1^2 & \cdots & a_1^n \\ a_2^1 & a_2^2 & \cdots & a_2^n \\ a_1^1 & a_1^2 & \cdots & a_1^n \end{Vmatrix}$$

vanish. Expanding each of the determinants along the last row the following set of equations is obtained:

$$a_1^i\, a_{12}^{jk} - a_1^j\, a_{12}^{ik} + a_1^k\, a_{12}^{ij} = 0, \qquad (i<j<k). \tag{20}$$

From equations (20) the cartesian equations of a plane in F_n can be readily deduced. It is only necessary to note that the coördinates of any point in the plane are proportional to the direction cosines of some line in it, hence (taking $i = 1$, $j = 2$),

$$X_1 P_{2k} - X_2 P_{1k} + X_k P_{12} = 0, \qquad k = 3, \cdots, n,$$

or,

$$P_{12} X_k = - P_{2k} X_1 + P_{1k} X_2, \tag{21}$$

where

$$P_{ik} = a_{12}^{ik}.$$

Equations (21) can be also written in the parametric form

$$P_{12} X_k = - P_{2k}\, t_1 + P_{1k}\, t_2, \tag{21a}$$

(adopting the convention $P_{rs} = - P_{sr}$ and $P_{rr} = 0$).

In order to express the direction cosines of two orthogonal lines in terms of the direction cosines of the plane that is determined by them, I shall add a set of equations similar to equations (20):

$$a_2^i\, a_{12}^{jk} - a_2^j\, a_{12}^{ik} + a_2^k\, a_{12}^{ij} = 0. \tag{22}$$

Letting $i = 1$ and $j = 2$ and using abbreviations as in equations (21) we shall have:

$$a_2^1 P_{2k} - a_2^2 P_{1k} + a_2^k P_{12} = 0. \tag{23}$$

There are $(2n - 3)$ independent parameters that determine directions of two orthogonal straight lines while a plane is determined by only $2(n - 2)$ essentially different parameters; hence, we can arbitrarily assign a value to some of the sought quantities, let us say

$$a_2^1 = 0. \tag{24}$$

If such be the case, equations (23) immediately furnish:

$$a_2^k = \frac{P_{1k}}{P_{12}} a_2^2. \tag{25}$$

Substituting these into

$$a_2^r a_2^r = 1$$

we have*

$$a_2^2 = \frac{P_{12}}{\sqrt{P_{1r} P_{1r}}} \tag{26}$$

(the sign of the square root can be evidently ascribed at will).

From equations (25) and (26) we have:

$$a_2^k = \frac{P_{1k}}{\sqrt{P_{1r} P_{1r}}}. \tag{27}$$

Now, I shall proceed to find the direction cosines a_1^i. First of all,

$$\begin{vmatrix} a_1^1 & a_1^2 \\ 0 & a_2^2 \end{vmatrix} = P_{12},$$

hence

$$a_1^1 = \frac{P_{12}}{a_2^2} = \sqrt{P_{1r} P_{1r}}. \tag{28}$$

Also,

$$\begin{vmatrix} a_1^2 & a_1^i \\ a_2^2 & a_2^i \end{vmatrix} = P_{2i},$$

consequently,

$$a_1^i = \frac{a_1^2 a_2^i - P_{2i}}{a_2^2}. \tag{29}$$

Substituting these values into

$$a_1^r a_2^r = 0$$

we have

$$a_2^r (a_1^2 a_2^r - P_{2r}) = 0, \tag{30}$$

* In what follows I shall assume $P_{rr} = 0$.

whence

$$a_1^2 = a_2^r P_{2r} = \frac{P_{1r} P_{2r}}{\sqrt{P_{1r} P_{1r}}}. \tag{31}$$

Substituting from (31) into (29) and making transformations we obtain:

$$a_1^i = \frac{P_{1r}(P_{2r} P_{1i} - P_{1r} P_{2i})}{P_{12} \sqrt{P_{1r} P_{1r}}}. \tag{32}$$

Equations (27) and (32) are true for all values of i and k from 1 to n.

8. Two planes whose direction cosines satisfy the equation

$$P_{ij} Q_{ij} = 0 \tag{33}$$

are said to be simply orthogonal. A set of planes passing through every pair of an orthogonal set of lines I shall call a simply orthogonal set of planes.

If the matrix of the direction cosines of the orthogonal set of lines be

$$\| a_k^i \|$$

then the matrix of the direction cosines of the simply orthogonal set of planes will be

$$\| a_{kl}^{ij} \| \qquad i < j, \; k < l.$$

The condition of orthogonality (33) of individual members of the set is verified. In fact, equations

$$\begin{aligned} a_{kl}^{ij} A_{rs}^{kl} &= 0, \quad \text{if} \quad i \neq r \quad \text{or} \quad j \neq s \\ &= 1, \quad \text{if} \quad i = r \text{ and } j = s \end{aligned} \tag{34}$$

are consequences of Laplace's theorem for expansion of determinants. Also it is known that

$$A_{rs}^{kl} = a_{kl}^{rs},$$

hence,

$$\begin{aligned} a_{kl}^{ij} a_{kl}^{rs} &= 0, \quad \text{if } i \neq r \quad \text{or} \quad j \neq s \\ &= 1, \quad \text{if } i = r \text{ and } j = s. \end{aligned} \tag{35}$$

Similarly,

$$(36)\qquad \begin{aligned} a^{kl}_{ij}\, a^{kl}_{rs} &= 0, \quad \text{if } i \neq r \text{ or } j \neq s \\ &= 1, \quad \text{if } i = r \text{ and } j = s. \end{aligned}$$

9. Suppose the equations of the metrical group

$$(37)\qquad y_i = a^r_i\, x_r$$

transform an orthogonal set of lines with the direction cosines forming the matrix $\| x^k_r \|$ into another orthogonal set $\| y^k_r \|$. The same transformation, then, changes $\| x^{kl}_{rs} \|$ the matrix of the simply orthogonal set of planes determined by the first set of lines into $\| y^{kl}_{ij} \|$ of the second set of planes.

Now,

$$y^{kl}_{ij} = \begin{vmatrix} y^k_i & y^l_i \\ y^k_i & y^l_j \end{vmatrix} = \begin{vmatrix} a^r_i & x^k_r, & a^r_i & x^l_r \\ a^s_j & x^k_s, & a^s_j & x^l_s \end{vmatrix};$$

whence

$$(38)\qquad y^{kl}_{ij} = \begin{vmatrix} a^r_i & a^s_i \\ a^r_j & a^s_j \end{vmatrix} \begin{vmatrix} x^k_r & x^l_r \\ x^k_s & x^l_s \end{vmatrix} = a^{rs}_{ij}\, x^{kl}_{rs}, \qquad \text{where } r < s.$$

Equations (38) transform one simply orthogonal set into another similar set. Any plane is transformed by equations (37) into another plane in accordance with

$$(39)\qquad Q_{ij} = a^{rs}_{ij}\, P_{rs}, \qquad r < s.$$

10. Two flats F_r and F_s are said to be absolutely orthogonal if every straight line in one is perpendicular to every straight line in the other.

Let two planes be determined respectively by two pairs of straight lines $\| a^1_i \|$, $\| a^2_i \|$ and $\| a^3_i \|$ and $\| a^4_i \|$. The planes will be absolutely orthogonal if any line $\| r_1 a^1_i + r_2 a^2_i \|$ in the first plane is perpendicular to any line $\| r_3 a^3_i + r_4 a^4_i \|$ in the second plane, i. e., if

$$(40)\qquad (r_1\, a^1_i + r_2\, a^2_i)\,(r_3\, a^3_i + r_4\, a^4_i) = 0$$

irrespectively of the values assumed by r_1, r_2, r_3 and r_4. This is obviously so if

$$
(41) \qquad \begin{aligned} a_i^1\, a_i^3 &= 0, \\ a_i^1\, a_i^4 &= 0, \\ a_i^2\, a_i^3 &= 0, \\ a_i^2\, a_i^4 &= 0. \end{aligned}
$$

Thus, equations (41) are sufficient conditions of absolute orthogonality of the two given planes and evidently, they are also necessary.

If we apply the definition of absolute orthogonality to the planes whose equations are given in a parametric form by (21a), we shall obtain the following conditions:

$$
(42) \qquad \begin{aligned} P_{1k}\, Q_{1k} &= 0, \\ P_{1k}\, Q_{2k} &= 0, \\ P_{2k}\, Q_{1k} &= 0, \\ P_{2k}\, Q_{2k} &= 0. \end{aligned}
$$

It is apparent, however, that these conditions can be applied only if neither P_{12} nor Q_{12} is equal to zero. If any of them is equal to zero, a similar set of equations will serve as a test of orthogonality. Also, it should be noted that absolute orthogonality is an invariant property under the metrical group.

11. A complete set of planes that have a point in common and are mutually absolutely orthogonal I shall call an absolutely orthogonal set of planes.

It is known that through any point of F_r in F_n there is one and only one F_{n-r} that is absolutely orthogonal to F_r.*

Take any one of $\infty^{(n-2)}$ planes through a point in F_n as the first member of an absolutely orthogonal set of planes. In F_{n-2} which is absolutely orthogonal to this plane take any plane as the second member of our set of planes, etc. Hence, there are $\infty^{2(n-2)+2(n-4)+\cdots}$ absolutely

* Woods, F. S., Higher Geometry, p. 417.

orthogonal sets of planes. If n is even the number is $\infty^{\frac{n(n-2)}{2}}$, if n is odd it is

$$\infty^{\frac{(n-1)^2}{2}}.$$

If n is even there are

$$\infty^{\frac{n}{2}}$$

rotations leaving invariant each member of a given absolutely orthogonal set of planes; if n is odd there are only

$$\infty^{\frac{n-1}{2}}$$

such rotations. In either case there will be

$$\infty^{\frac{n(n-1)}{2}}$$

rotations leaving invariant each member of some absolutely orthogonal set of planes. These are all of the rotations in F_n. Therefore, *the most general rotation in F_n can be decomposed into $E\left(\frac{n}{2}\right)$ simple rotations in the ordinary planes of some appropriately chosen absolutely orthogonal set of planes. This decomposition is in general unique.* In the next chapter I shall prove the theorem analytically and shall give equations for determining the invariant planes as well as angles of simple rotations.*

Also it should be noticed that in an odd flat F_{2n-1} for every rotation there will be an absolutely invariant (every point on it is invariant) straight line, i. e., an "axis" of rotation, namely the line that is perpendicular to the invariant absolutely orthogonal set of planes.

---

* In case $n = 4$ this theorem was proved analytically by F. N. Cole in his paper on Rotations in Space of Four Dimensions, Amer. Journ. of Math., vol. XII (1890), pp. 191–210. His method cannot be conveniently extended, however, to any F_n.

CHAPTER III

12. A flat N_m is said to be null if its differential element for the length identically vanishes.

To determine null lines through a point in F_n assume their equations in the form

$$x_r = i a_r x_n \qquad r = 1, \cdots, n-1 \tag{43}$$

and substitute into Monge's equation

$$dx_1^2 + dx_2^2 + \cdots + dx_n^2 = 0. \tag{44}$$

We have,

$$a_1^2 + a_2^2 + \cdots + a_{n-1}^2 = 1. \tag{45}$$

Hence, the *null lines through a point in F_n can be put into one to one correspondence with the ordinary directions in F_{n-1} (opposite directions of the same straight line correspond to conjugate null lines). Ratios of the direction cosines of a null line to some one cosine of its set are respectively equal to the direction cosines of the corresponding ordinary line multiplied by i. The sum of the squares of the direction cosines of a null line is equal to zero.*

In general m-dimensional null flates N_m through a point in F_n are in one to one correspondence with the orthogonal sets of "m" ordinary directions in F_{n-m}.

In fact, let

$$x_r = i a_r^1 x_{m+1} + i a_r^2 x_{m+2} + \cdots + i a_r^{n-m} x_n \qquad r = 1, \cdots, m \tag{46}$$

be the equations of a null flat of N_m. Substituting the expressions for x_r from equations (46) into Monge's equation we obtain the following conditions that must be satisfied by the coefficients a_r^s:

$$
(47) \qquad \begin{aligned} a_r^s a_r^t &= 0 \text{ if } s \neq t \qquad r = 1, \cdots, m \\ &= 1 \text{ if } s = t \quad s, t = 1, \cdots, n-m. \end{aligned}
$$

This proves the theorem. An immediate corollary of it is: *the number of dimensions of a null subflat N_m in F_n cannot exceed $\frac{n}{2}$.** The statement follows from the inequality

$$m \leqq n - m,$$

hence,

$$(48) \qquad m \leqq \frac{n}{2}.$$

13. In F_3 and higher manifolds there are curved null subspaces. Thus, in F_3 we have null curves given by

$$
(49) \qquad \begin{aligned} x_1 &= tf''(t) - f'(t) + \text{const.}, \\ 2ix_2 &= [t^2 f''(t) - 2tf'(t) + 2f(t)] + f''(t) + \text{const.}, \\ 2x_3 &= [t^2 f''(t) - 2tf'(t) + 2f(t)] + f''(t) + \text{const.}, \end{aligned}
$$

where $f(t)$ is an arbitrary function of t excluding polynomials of the second or lower degree.†

I shall derive similar equations for null curves in F_4. Monge's equation in this case can be written

$$dx_3^2 + dx_4^2 = -(dx_1^2 + dx_2^2)$$

or

$$
\begin{aligned} &(dx_3 + i\,dx_4)(dx_3 - i\,dx_4) \\ &\qquad = -(dx_1 + i\,dx_2)(dx_1 - i\,dx_2). \end{aligned}
$$

Now, if the equations of a null curve are

$$x_i = f_i(\omega) \qquad i = 1, \cdots, 4$$

---

* Kasner, E., Null geometry, Bull. Amer. Math. Soc., vol. XXXI (1925), p. 293. Lense, J., "Über ametrische Mannigfaltigkeiten", Jahresbericht der Deutschen Mathematiker-Vereinigung, 35. Band, 9.–12. Heft, pp. 280–294 (1926).

† Weierstrass, K., Monatsberichte der Berliner Akademie (1866), p. 619. Blaschke, W., Vorlesungen über Differentialgeometrie, p. 30.

then

$$dx_3 + i\,dx_4 = t(dx_1 + i\,dx_2),$$
$$dx_3 - i\,dx_4 = -\frac{1}{t}(dx_1 - i\,dx_2),$$

where t is an arbitrary function of ω. From these equations we obtain immediately

$$(50)\quad \begin{aligned} 2x_3 &= \int \frac{t^2-1}{t}\,dx_1 + i\int \frac{t^2+1}{t}\,dx_2 + \text{const.},\\ 2ix_4 &= \int \frac{t^2+1}{t}\,dx_1 + i\int \frac{t^2-1}{t}\,dx_2 + \text{const.},\end{aligned}$$

where x_1 and x_2 are arbitrary functions of ω. Let us choose t as a parameter; then ω and hence x_1 and x_2 will be arbitrary functions of t. Put

$$(51)\quad \begin{aligned} dx_1 &= tf_1'''(t)\,dt,\\ dx_2 &= tf_2'''(t)\,dt,\end{aligned}$$

where f_1 and f_2 are arbitrary functions of t excluding polynomials of the second or lower degrees.

Integrating equations (50) and (51) we have:

$$(52)\quad \begin{aligned} x_1 &= tf_1''(t) - f_1'(t) + \text{const.},\\ x_2 &= tf_2''(t) - f_2'(t) + \text{const.},\\ 2x_3 &= F(t) - \frac{d^2}{dt^2}[f_1(t) - if_2(t)] + \text{const.},\\ 2ix_4 &= F(t) + \frac{d^2}{dt^2}[f_1(t) - if_2(t)] + \text{const.},\end{aligned}$$

where,

$$F(t) = \left[t^2\frac{d^2}{dt^2} - 2t\frac{d}{dt} + 2\right][f_1(t) + if_2(t)].$$

It seems that in higher F_n equations of null curves cannot be put in a form free from quadratures.

Null curves in F_n I shall give by

$$
(53)\quad
\begin{aligned}
x_2 &= i\int \cos f_1 \cos f_2 \cdots \cos f_{n-2}\, dx_1 + \text{const.},\\
x_3 &= i\int \sin f_1 \cos f_2 \cdots \cos f_{n-2}\, dx_1 + \text{const.},\\
x_4 &= i\int \sin f_2 \cos f_3 \cdots \cos f_{n-2}\, dx_1 + \text{const.},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
x_{n-1} &= i\int \sin f_{n-3} \cos f_{n-2}\, dx_1 + \text{const.},\\
x_n &= i\int \sin f_{n-2}\, dx_1 \qquad\qquad + \text{const.},
\end{aligned}
$$

where $f_1, f_2, f_3, \cdots, f_{n-2}$ are arbitrary functions of x_1. This can be verified by direct substitution into Monge's equation.

Some of the null surfaces S_{n+1} in F_{2n+1} are given by

$$
(54)\quad
\begin{aligned}
x_{n+1} &= \pm i x_1,\\
x_{n+2} &= \pm i x_2,\\
&\cdots\cdots\\
x_{2n-1} &= \pm i x_{n-1},\\
x_n &= t f''(t) - f'(t),\\
x_{2n} &= [t^2 f''(t) - 2t f'(t) + 2f(t)] + f''(t),\\
x_{2n+1} &= [t^2 f''(t) - 2t f'(t) + 2f(t)] - f''(t),
\end{aligned}
$$

where $f(t)$ is an arbitrary function of the parameter t.

14. I shall determine now the straight lines in F_n that remain invariant under a rotation in that space. Let $x_1, x_2, \cdots, x_n$ be the direction cosines of an invariant straight line if there is such. Then after the rotation the new values of the direction cosines must be proportional to the corresponding old values. Hence, a set of equations

$$
(55)\qquad kx_r = a_r^s x_s
$$

(where k is the coefficient of proportionality) or

$$
(56)\quad
\begin{aligned}
(a_1^1 - k)x_1 + a_1^2 x_2 + a_1^3 x_3 + \cdots + a_1^n x_n &= 0,\\
a_2^1 x_1 + (a_2^2 - k)x_2 + a_2^3 x_3 + \cdots + a_2^n x_n &= 0,\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots &\\
a_n^1 x_1 + a_n^2 x_2 + a_n^3 x_3 + \cdots + (a_n^n - k)x_n &= 0.
\end{aligned}
$$

The set of equations (56) will have a non-trivial solution if the determinant of its coefficients vanishes, i. e., if

$$
(57)\quad
\begin{vmatrix}
a_1^1 - k & a_1^2 & a_1^3 & \cdots & a_1^n\\
a_2^1 & a_2^2 - k & a_2^3 & \cdots & a_2^n\\
\dots & \dots & \dots & \dots & \dots\\
a_n^1 & a_n^2 & a_n^3 & \cdots & a_n^n - k
\end{vmatrix} = 0.
$$

This equation is of nth degree and has n solutions (obviously none of which can ever be equal to zero). According to the theorem of Brioschi* equation (57) can have no real roots different from ± 1. If n is even in general there are no real roots and there is only one real root $k = 1$ if n is odd. In the latter case we have a line of invariant points.

In general, the first minors of the determinant (57) are not all equal to zero. This establishes the theorem: *Every rotation in F_n leaves n straight lines and in general only n straight lines invariant. If n is odd one of these lines is real and absolutely invariant. The remaining lines are conjugate imaginary and moreover they are null.* The last clause of the theorem may need further explanation. If an invariant line is ordinary, the coefficient of proportionality k cannot have values different from ± 1 as immediately seen if equations (55) are squared and then added together.

Each conjugate pair of lines determines a real plane. Thus we have the theorem: *Every rotation in F_n leaves $E\left(\frac{n}{2}\right)$ real planes invariant $\left(\frac{n}{2}\text{ if } n \text{ is even}, \frac{n-1}{2} \text{ if odd}\right)$,*

---

* Kowalewski, G., p. 163.

and hence it consists of $E\left(\frac{n}{2}\right)$ *successive rotations in invariant planes through the center of rotation.*

The total number of invariant planes is $\frac{n(n-1)}{2}$ *out of which in case* n *is even* $\frac{n}{2}$ *are ordinary and real and* $\frac{n(n-2)}{2}$ *are null; in case* n *is odd there are* $\frac{n-1}{2}$ *ordinary and real,* $(n-1)$ *aeolotropic,* and* $\frac{(n-1)(n-3)}{2}$ *null planes.*

In the above I have assumed that the coefficients of the metrical group are real. The preceding theorems can be extended to the case when they are complex with the exception that the ordinary invariant planes do not have to be real.

Now, I shall inquire into the nature of k and the equation (57).

Take a plane rotation through an angle θ.

$$\begin{aligned} y_1 &= x_1 \cos\theta - x_2 \sin\theta, \\ y_2 &= x_1 \sin\theta + x_2 \cos\theta. \end{aligned} \tag{58}$$

The invariant lines under this rotation are:

$$x_2 = \pm i x_1. \tag{59}$$

Their direction cosines $(1, i)$ and $(1, -i)$ are changed by the transformation (58) into $(e^{-i\theta}, i e^{-i\theta})$ and $(e^{i\theta}, i e^{i\theta})$. Thus, I note that rotations in a plane through the angle θ multiply the direction cosines of its invariant lines respectively by $e^{-i\theta}$ and $e^{i\theta}$. Rotations in planes that are absolutely perpendicular to the given plane leave every point and hence every line absolutely invariant; therefore, they do not change direction cosines. Thus, *the complex roots of*

* A plane determined by an ordinary line and a null line is called aeolotropic.

equation (57) *are* $e^{\pm i\theta_k}$ *where* $\theta_k\left(k = 1, \cdots, E\left(\frac{n}{2}\right)\right)$ *are the angles of component simple rotations.* Equation (57) may be called "rotational extent" equation as its solution enables us to find the "extent" of simple rotations that make the compound rotation.

If we develop the determinant in (57) we shall obtain:*

$$(60)\quad k^n - s_1 k^{n-1} + s_2 k^{n-2} - \cdots + (-1)^{n-1} s_{n-1} k + (-1)^n = 0,$$

where s_r is the sum of the rth order principal minors of the determinant $|a_k^i|$, i. e.,

$$(61)\quad \begin{aligned} s_1 &= a_1^1 + a_2^2 + \cdots + a_n^n, \\ s_2 &= a_{12}^{12} + a_{13}^{13} + a_{14}^{14} + \cdots + a_{23}^{23} + \cdots + a_{n-1,n}^{n-1,n}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \text{ etc.} \end{aligned}$$

Moreover,

$$(62)\quad s_1 = s_{n-1},$$

and, in general

$$s_r = s_{n-r}.$$

Therefore, equation (60) is symmetrical if n is even; if n is odd it will be symmetrical after the factor $(k-1)$ is removed. If n is even in accordance with the above

$$(63)\quad \begin{aligned} s_1 &= 2(\cos\theta_1 + \cos\theta_2 + \cdots + \cos\theta_m) \qquad \left(m = \frac{n}{2}\right), \\ s_2 &= m + 4(\cos\theta_1\cos\theta_2 + \cos\theta_1\cos\theta_3 + \cdots \\ &\qquad + \cos\theta_{m-1}\cos\theta_m), \text{ etc.} \end{aligned}$$

Also if

$$(64)\quad k + \frac{1}{k} = 2t$$

and the appropriate substitution into equation (60) is made a new equation of degree $\frac{n}{2}$ is obtained whose roots are the cosines of the angles of simple rotations.

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* Kowalewski, p. 126.

Similar transformations can be obtained if n is odd. In this case,

$$
(65)\quad
\begin{aligned}
s_1 &= 2(\cos\theta_1 + \cos\theta_2 + \cdots + \cos\theta_m) + 1,\\
s_2 &= 4(\cos\theta_1\cos\theta_2 + \cos\theta_1\cos\theta_3 + \cdots + \cos\theta_{m-1}\cos\theta_m)\\
&\quad + 2(\cos\theta_1 + \cos\theta_2 + \cdots + \cos\theta_m) + \frac{n-1}{2}, \text{ etc.},\\
&\text{where } m = \frac{n-1}{2}.
\end{aligned}
$$

The coefficients $s_1, s_2, \cdots, s_{n-1}$ are obviously independent of the choice of the rectangular coördinate system; therefore, they are invariant under cogredient orthogonal transformations of the equations of the metrical group.

If one of the invariant planes is fixed absolutely (i.e. every point in it is fixed) then the angle of rotation in that plane is zero and equation (57) has 1 as its double root. If r planes are absolutely invariant equation (57) has 1 as its multiple root of order $2r$.

If simple rotations in r planes are upon equal angles equation (57) has $2r$ equal roots. If these invariant planes are to be determined from equations (56) and (57) the problem will be indeterminate and there will be an infinite number of invariant planes. In the next section I shall discuss this exceptional case.

15. I shall confine myself to F_4. To examine the nature of the case of "equal" simple rotations I shall choose coördinate axes in such a way that the planes x_{12} and x_{34} are invariant; hence, the equations of rotation will be

$$
(66)\quad
\begin{aligned}
y_1 &= cx_1 - sx_2,\\
y_2 &= sx_1 + cx_2,\\
y_3 &= cx_3 - sx_4,\\
y_4 &= sx_3 + cx_4,
\end{aligned}
$$

where c and s are respectively the cosine and sine of the angle of rotation.

Equation (57) becomes,

$$
(67)\quad [(c-k)^2 + s^2]^2 = 0,
$$

and

$$k = c \pm is.$$

Equations (56) degenerate into

$$(68)\qquad \begin{aligned} (c-k)\,x_1 - sx_2 &= 0,\\ sx_1 + (c-k)\,x_2 &= 0,\\ (c-k)\,x_3 - sx_4 &= 0,\\ sx_3 + (c-k)\,x_4 &= 0, \end{aligned}$$

and their solutions are two pencils of lines

$$(69)\qquad \begin{aligned} x_2 &= ix_1, & \text{and}\quad x_2 &= -ix_1,\\ x_3 &= m'\,x_1, & x_3 &= m''x_1,\\ x_4 &= im'x_1, & x_4 &= im''x_1. \end{aligned}$$

These lines determine a two parameter family of invariant planes:

$$(70)\qquad \begin{aligned} x_3 &= \frac{m'+m''}{2}\,x_1 + \frac{m'-m''}{2i}\,x_2,\\ x_4 &= -\frac{m'-m''}{2i}\,x_1 + \frac{m'+m''}{2}\,x_2, \end{aligned}$$

or

$$(71)\qquad \begin{aligned} x_3 &= m_1 x_1 - m_2 x_2,\\ x_4 &= m_2 x_1 + m_1 x_1. \end{aligned}$$

Therefore, there are ∞^2 of real planes that remain invariant under the transformation (66).

Any line through the origin in the plane (71)

$$(72)\qquad \frac{x_1}{1} = \frac{x_2}{k} = \frac{x_3}{m_1 - m_2 k} = \frac{x_4}{m_2 + m_1 k}$$

is transformed into

$$(73)\qquad \begin{aligned} \frac{x_1}{c-ks} = \frac{x_2}{kc+s} &= \frac{x_3}{m_1(c-ks) - m_2(kc+s)}\\ &= \frac{x_4}{m_1(kc+s) + m_2(c-ks)}. \end{aligned}$$

The cosine of the angle between the two lines is

$$\cos\theta = c, \tag{74}$$

i. e., every line through the origin in any of the invariant planes rotates through the same angle.

The planes that are absolutely orthogonal with the corresponding members of the family (71) are

$$\begin{aligned} (m_1^2 + m_2^2)x_3 &= -m_1x_1 + m_2x_2, \\ (m_1^2 + m_2^2)x_4 &= -m_2x_1 - m_1x_2. \end{aligned} \tag{75}$$

This family of planes is identical with (71) and hence *we can choose* ∞^2 *pairs of absolutely orthogonal planes in which the compound rotation* (66) *can be resolved into a succession of two simple rotations.*

There are only ∞^3 (in F_4) rotations that leave each member of some two parameter family of planes invariant.

Exceptional cases in higher F_n can be similarly treated.

CHAPTER IV

16. Choose a null n-flat N_n in F_{2n}

$$x_{n+r} = ix_r, \qquad r = 1, \cdots, n. \tag{76}$$

Under what conditions does the metrical group in F_{2n} leave N_n invariant? Let the transformations be

$$y_k = a_k^s x_s \qquad k, s = 1, \cdots, 2n \tag{77}$$

and equations (76) be transformed into

$$y_{n+r} = iy_r, \qquad r = 1, \cdots, n. \tag{78}$$

Substitute from (76) into (77)

$$y_k = (a_k^r + i a_k^{n+r})\, x_r \quad r = 1, \cdots, n;\ k = 1, \cdots, 2n, \tag{79}$$

which on account of (78) are resolved into two systems

$$\begin{aligned} y_k &= (a_k^r + i a_k^{n+r})\, x_r \quad r, k = 1, \cdots, n, \\ y_k &= (-i a_{n+k}^r + a_{n+k}^{n+r})\, x_r, \end{aligned} \tag{80}$$

which must be identical, hence

$$a_k^r + i a_k^{n+r} = a_{n+k}^{n+r} - i a_{n+k}^r. \tag{81}$$

I shall confine myself to the case of the "real" metrical group in F_{2n}. Then from (81)

$$a_{n+k}^{n+r} = a_k^r, \qquad a_{n+k}^r = -a_k^{n+r}. \tag{82}$$

If the conditions (82) are satisfied N_n is transformed into itself and every point in it is transformed in accordance with the set of equations:

(83) $$y_k = A_k^r x_r \qquad r = 1, \cdots, n,$$

where

(84) $$A_k^r = a_k^r + i a_k^{n+r}.$$

The problem now is to find the relations between the coefficients A_k^r.

Transformations (77) are orthogonal; the conditions of orthogonality can be written in the form:

(85) $$\begin{aligned} a_k^s a_l^s &= 1, \quad \text{if } l = k; \quad s = 1, \cdots, 2n, \\ &= 0, \quad \text{if } l \neq k; \; k, l = 1, \cdots, n, \\ a_k^s a_{n+l}^s &= 0. \end{aligned}$$

When $l = k$ we have immediately

(86) $$A_k^r \bar{A}_l^r = 1,$$

where $\bar{A}_l^r$ is the conjugate of A_l^r. Otherwise, multiply the third set of equations by i and add to the corresponding equations of the second set:

$$a_k^s a_l^s + i a_k^s a_{n+l}^s = 0 \qquad s = 1, \cdots, 2n.$$

On account of (82) and (84) we obtain:

(87) $$A_k^r \bar{A}_l^r = 0 \qquad r = 1, \cdots, n; \; k, l = 1, \cdots, n; \; k \neq l.$$

Combining (86) and (87):

(88) $$\begin{aligned} A_k^r \bar{A}_l^r &= 1, \quad \text{if } k = l, \\ &= 0, \quad \text{if } k \neq l. \end{aligned}$$

Similarly, the equations

(89) $$\begin{aligned} A_r^k \bar{A}_r^l &= 1, \quad \text{if } k = l, \\ &= 0, \quad \text{if } k \neq l \end{aligned}$$

are true.

I shall call the group of transformations (83), the coefficients A_k^r being subject to the conditions (88), the genera-

lized orthogonal group.* The determinant of its coefficients I shall call the generalized orthogonant to the derivation of some properties of which I pass.

17. Let

$$D = \begin{vmatrix} A_1^1 & A_1^2 & A_1^3 & \cdots & A_1^n \\ A_2^1 & A_2^2 & A_2^3 & \cdots & A_2^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ A_n^1 & A_n^2 & A_n^3 & \cdots & A_n^n \end{vmatrix} \tag{90}$$

be a generalized orthogonant.

Multiply every column of D respectively by $\overline{A}_1^1, \overline{A}_1^2, \cdots, \overline{A}_1^n$ and add them to the first:

$$D = \frac{1}{\overline{A}_1^1 \overline{A}_1^2 \cdots \overline{A}_1^n} \begin{vmatrix} A_1^1 \overline{A}_1^1, & A_1^2 \overline{A}_1^2, & \cdots, & A_1^n \overline{A}_1^n \\ A_2^1 \overline{A}_1^1, & A_2^2 \overline{A}_1^2, & \cdots, & A_2^n \overline{A}_1^n \\ \cdot & \cdot & \cdot & \cdot \\ A_n^1 \overline{A}_1^1, & A_n^2 \overline{A}_1^2, & \cdots, & A_n^n \overline{A}_1^n \end{vmatrix}$$

$$= \frac{1}{\overline{A}_1^1 \overline{A}_1^2 \cdots \overline{A}_1^n} \begin{vmatrix} 1, & A_1^2 \overline{A}_1^2, & \cdots, & A_1^n \overline{A}_1^n \\ 0, & A_2^2 \overline{A}_1^2, & \cdots, & A_2^n \overline{A}_1^n \\ \cdot & \cdot & \cdot & \cdot \\ 0, & A_n^2 \overline{A}_1^2, & \cdots, & A_n^n \overline{A}_1^n \end{vmatrix} = \frac{1}{\overline{A}_1^1} B_1^1,$$

where B_1^1 is the cofactor of A_1^1.

Hence,

$$B_1^1 = D\overline{A}_1^1,$$

and in general

$$B_k^r = D\overline{A}_r^k, \tag{91}$$

i.e., *the cofactor of any element of a generalized orthogonant is equal to the conjugate of that element multiplied by D.*

Consider the reciprocal determinant of D. It is equal to D^{n-1}; on the other hand on account of (91) it is equal to $D^n \overline{D}$ ($\overline{D}$ being conjugate of D); hence,

* The term quasi-orthogonal has its advantages. There is an objection to its use, however. The ordinary orthogonal group with real coefficients is a special case of the generalized orthogonal group.

$$D\bar{D} = 1, \tag{92}$$

i. e, *the absolute value of a generalized orthogonant is* 1.

The equation (91) can be then, written:

$$A_r^k = D\bar{B}_k^r. \tag{93}$$

This can be generalized. Namely, *every minor of a generalized orthogonant is equal to the conjugate of its cofactor multiplied by D.*

To prove the theorem I shall use the following property of reciprocal determinants:* Let D be a determinant different from zero and d its reciprocal determinant. If M is a minor of kth order of D and m is the corresponding minor of d, then m is equal to the cofactor of M multiplied by D^{k-1}.

Hence,

$$\bar{D}^k \begin{vmatrix} A_1^1 & A_1^2 & \cdots & A_1^k \\ A_2^1 & A_2^2 & \cdots & A_2^k \\ \cdot & \cdot & \cdot & \cdot \\ A_k^1 & A_k^2 & \cdots & A_k^k \end{vmatrix} = \begin{vmatrix} \bar{D}A_1^1 & \bar{D}A_1^2 & \cdots & \bar{D}A_1^k \\ \bar{D}A_2^1 & \bar{D}A_2^2 & \cdots & \bar{D}A_2^k \\ \cdot & \cdot & \cdot & \cdot \\ \bar{D}A_k^1 & \bar{D}A_k^2 & \cdots & DA_k^k \end{vmatrix}$$

$$= \begin{vmatrix} \bar{B}_1^1 & \bar{B}_2^1 & \cdots & \bar{B}_k^1 \\ \bar{B}_1^2 & \bar{B}_2^2 & \cdots & \bar{B}_k^2 \\ \cdot & \cdot & \cdot & \cdot \\ \bar{B}_1^k & \bar{B}_2^k & \cdots & \bar{B}_k^k \end{vmatrix} = \bar{D}^{k-1} \begin{vmatrix} \bar{A}_{k+1}^{k+1} & \bar{A}_{k+1}^{k+2} & \cdots & \bar{A}_{k+1}^n \\ \bar{A}_{k+2}^{k+1} & \bar{A}_{k+2}^{k+2} & \cdots & \bar{A}_{k+2}^n \\ \cdot & \cdot & \cdot & \cdot \\ \bar{A}_n^{k+1} & \bar{A}_n^{k+2} & \cdots & \bar{A}_n^n \end{vmatrix}$$

therefore,

$$A_{12\ldots k}^{12\ldots k} = D\bar{B}_{12\ldots k}^{12\ldots k} \tag{94}$$

and the theorem is proved. The transformations inverse of (83) are

$$Dx_k = B_k^r y_r \quad \text{or} \quad x_k = \bar{A}_r^k y_r \tag{95}$$

on account of (91).

The coefficients of the generalized orthogonal group are not all independent. They contain $2n^2$ real parameters

* Kowalewski, p. 80.

among which there are $n+n(n-1)=n^2$ relations; hence, there are only n^2 real independent parameters. I shall derive expressions for the coefficients of the group in terms of such independent parameters. These expressions are similar to those obtained by A. Cayley and his method holds with only slight modifications in the more general case.

I shall assume a determinant

$$C = \begin{vmatrix} c_1^1 & c_1^2 & \cdots & c_1^n \\ c_2^1 & c_2^2 & \cdots & c_2^n \\ \cdot & \cdot & \cdot & \cdot \\ c_n^1 & c_n^2 & \cdots & c_n^n \end{vmatrix} \tag{96}$$

where

$$c_s^r + \overline{c}_r^s = 0 \quad \text{if} \quad r \neq s,$$

(97) and

$$c_1^1 + \overline{c}_1^1 = c_2^2 + \overline{c}_2^2 = \cdots = c_n^n + \overline{c}_n^n = c.$$

Hence, the quantities c_s^r depend upon n^2+1 real parameters, one of which is non-essential in the final results.

Assume two systems of equations

$$x_k = c_k^r u_r, \tag{98}$$

and

$$y_k = \overline{c}_r^k u_r. \tag{99}$$

Adding the corresponding equations in (98) and (99):

$$x_k + y_k = c u_k. \tag{100}$$

Solve (98) for u_k:

$$C u_k = C_k^r x_r \tag{101}$$

and substitute in it u_k from (100):

$$\begin{aligned} Cy_1 &= (cC_1^1 - C)x_1 + cC_1^2 x_2 + \cdots + cC_1^n x_n, \\ Cy_2 &= cC_2^1 x_1 + (cC_2^2 - C)x_2 + \cdots + cC_2^n x_n. \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ Cy_n &= cC_n^1 x_1 + cC_n^2 x_2 \qquad + \cdots + (cC_n^n - C)x_n. \end{aligned} \tag{102}$$

Solve (99) for u_k:

$$\bar{C}u_k = \bar{C}_r^k y_r \tag{103}$$

and substitute in it u_k from (100):

$$\begin{aligned}
\bar{C}x_1 &= (c\bar{C}_1^1 - \bar{C})y_1 + c\bar{C}_2^1 y_2 + \cdots + c\bar{C}_n^1 y_n, \\
\bar{C}x_2 &= c\bar{C}_1^2 y_1 + (c\bar{C}_2^2 - \bar{C})y_2 + \cdots + c\bar{C}_n^2 y_n, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\bar{C}x_n &= c\bar{C}_1^n y_1 + c\bar{C}_2^n y_2 \qquad + \cdots + c\bar{C}_n^n y_n.
\end{aligned} \tag{104}$$

Let

$$\frac{cC_r^s}{C} = A_r^s, \qquad r \neq s$$

(105) and

$$\frac{cC_1^1 - C}{C} = A_1^1, \ldots, \frac{cC_n^n - C}{C} = A_n^n,$$

then the transformations (102) and (104) are exactly like (83) and (95). Hence, they must be generalized orthogonal.

18. In discussion of algebraic and differential invariants of the generalized orthogonal group at first I shall limit myself to the two-dimensional case.

In what follows I shall designate the coördinates of a point by x and y and shall write the equations of transformation in the form

$$\begin{aligned} x' &= ax + by, \\ y' &= cx + dy, \end{aligned} \qquad \text{and} \qquad D = ad - bc, \tag{106}$$

where the coefficients satisfy the following conditions:*

$$\begin{aligned}
a\bar{a} + b\bar{b} &= 1, & \text{and} \qquad a\bar{a} + c\bar{c} &= 1, \\
c\bar{c} + d\bar{d} &= 1, & b\bar{b} + d\bar{d} &= 1, \\
a\bar{b} + c\bar{d} &= 0, & a\bar{c} + b\bar{d} &= 0, \\
\bar{a}b + \bar{c}d &= 0, & \bar{a}c + \bar{b}d &= 0.
\end{aligned} \tag{107}$$

Then the following invariants exist:

$$x'\bar{x}' + y'\bar{y}' = x\bar{x} + y\bar{y}, \tag{108}$$

$$x'\bar{x}_1' + y'\bar{y}_1' = x\bar{x}_1 + y\bar{y}_1, \tag{109}$$

* Only three of which are independent.

$$\bar{x}' x_1' + y' y_1' = \bar{x} x_1 + \bar{y} y_1, \tag{110}$$

$$x' y_1' - y' x_1' = D(x y_1 - y x_1), \tag{111}$$

and hence

$$(x' y_1' - y' x_1')(\bar{x}' \bar{y}_1' - \bar{y}' \bar{x}_1') = (x y_1 - y x_1)(\bar{x} \bar{y}_1 - \bar{y} \bar{x}_1). \tag{112}$$

For instance,

$$\begin{aligned} x' \bar{x}' + y' \bar{y}' &= (ax + by)(\bar{a}\bar{x} + \bar{b}\bar{y}) + (cx + dy)(\bar{c}\bar{x} + \bar{d}\bar{y}) \\ &= (a\bar{a} + c\bar{c}) x\bar{x} + (a\bar{b} + c\bar{d}) x\bar{y} + (\bar{a}b + \bar{c}d)\bar{x}y \\ &\qquad + (b\bar{b} + d\bar{d}) y\bar{y} \\ &= x\bar{x} + y\bar{y}. \end{aligned}$$

The remaining invariants are verified similarly.

From (109 and (110) the following invariants can be immediately deduced

$$\frac{1}{2}(x\bar{x}_1 + y\bar{y}_1 + \bar{x}x_1 + \bar{y}y_1) \tag{113}$$

and

$$\sqrt{(x\bar{x}_1 + y\bar{y}_1)(\bar{x}x_1 + \bar{y}y_1)}. \tag{114}$$

Then, from (108), (112), (113) and (114):

$$\cos\theta = \sqrt{\frac{(x\bar{x}_1 + y\bar{y}_1)(\bar{x}x_1 + \bar{y}y_1)}{(x\bar{x} + y\bar{y})(x_1\bar{x}_1 + y_1\bar{y}_1)}} \tag{115}$$

and

$$\cos\Phi = \frac{x\bar{x}_1 + y\bar{y}_1 + \bar{x}x_1 + \bar{y}y_1}{2\sqrt{(x\bar{x} + y\bar{y})(x_1\bar{x}_1 + y_1\bar{y}_1)}}, \tag{116}$$

where θ, the angle between the lines from the origin to the points (x, y) and (x_1, y_1), is defined by (115) and Φ, the angle between "linear threads" on those lines, is defined by (116).

To justify these definitions I shall take two lines

$$y = mx \quad \text{and} \quad y_1 = nx_1 \tag{117}$$

and compute $\cos\theta$ and $\cos\Phi$.

$$\begin{aligned} \cos\theta &= \sqrt{\frac{(1 + m\bar{n})(1 + \bar{m}n) x\bar{x}x_1\bar{x}_1}{(1 + m\bar{m})(1 + n\bar{n}) x\bar{x}x_1\bar{x}_1}} \\ &= \sqrt{\frac{(1 + m\bar{n})(1 + \bar{m}n)}{(1 + m\bar{m})(1 + n\bar{n})}}. \end{aligned} \tag{118}$$

$$(119) \qquad \cos\Phi = \frac{(1+m\bar{n})\,x\bar{x}_1 + (1+\bar{m}n)\,\bar{x}x_1}{2\sqrt{(1+m\bar{m})\,(1+n\bar{n})\,x\bar{x}x_1\bar{x}_1}}.$$

The coördinates of the line (117)

$$(120) \qquad \begin{aligned} x &= re^{i\alpha} \\ y &= mre^{i\alpha} \end{aligned}$$

depend upon two parameters. If α is constant (120) will define a single parameter sequence of points which I shall call "a linear thread" on the line $y = mx$.

Take another thread on the line $y = nx$:

$$(121) \qquad \begin{aligned} x &= Re^{i\beta} \\ y &= nRe^{i\beta}. \end{aligned}$$

Make substitutions from (120) and (121) into (119):

$$(122) \quad \cos\Phi = \frac{(1+m\bar{n})\,e^{i(\alpha-\beta)} + (1+\bar{m}n)\,e^{-i(\alpha-\beta)}}{2\sqrt{(1+m\bar{m})\,(1+n\bar{n})}},$$

i. e., $\cos\Phi$ depends upon "directions" of threads as well as upon directions of lines themselves.

Take two threads upon the same line, i. e., let $m = n$ in (122); then

$$(123) \qquad \cos\Phi = \frac{e^{i(\alpha-\beta)} + e^{-i(\alpha-\beta)}}{2} = \cos(\alpha-\beta),$$

and

$$(124) \qquad \Phi = \alpha - \beta.$$

Let

$$m = Pe^{iA}$$

and

$$n = Qe^{iB}$$

then

$$(125) \quad \cos\theta = \frac{\sqrt{1+2PQ\cos(A-B)+P^2Q^2}}{\sqrt{(1+P^2)\,(1+Q^2)}}$$

and

$$(126)\quad \begin{aligned}\cos\Phi &= \frac{\cos(\alpha-\beta)+PQ\cos(A-B+\alpha-\beta)}{\sqrt{(1+P^2)(1+Q^2)}}\\ &= \frac{[1+PQ\cos(A-B)]\cos(\alpha-\beta)-PQ\sin(A-B)\sin(\alpha-\beta)}{\sqrt{(1+P^2)(1+Q^2)}}.\end{aligned}$$

Considering the two lines given, i. e., m and n fixed and inspecting the form of (126) I conclude that there should be a value $(\alpha-\beta)$ for which Φ is minimum. In fact, taking the derivative of $\cos\Phi$ with respect to $\alpha-\beta$ and equating it to zero:

$$[1+PQ\cos(A-B)]\sin(\alpha-\beta)+PQ\sin(A-B)\sin(\alpha-\beta)=0$$

hence

$$\tan(\alpha-\beta) = -\frac{PQ\sin(A-B)}{1+PQ\cos(A-B)},$$

also

$$(127)\quad \begin{aligned}\sin(\alpha-\beta) &= -\frac{PQ\sin(A-B)}{\sqrt{1+2PQ\cos(A-B)+P^2Q^2}},\\ \cos(\alpha-\beta) &= -\frac{1+PQ\cos(A-B)}{\sqrt{1+2PQ\cos(A-B)+P^2Q^2}}.\end{aligned}$$

Substituting these values into (126) we have for the minimum angle Φ:

$$(128)\quad \cos\Phi = \frac{\sqrt{1+2PQ\cos(A-B)+P^2Q^2}}{\sqrt{(1+P^2)(1+Q^2)}},$$

i. e., the angle between two straight lines is equal to the minimum value of the angle between their linear threads. The angle $(\alpha-\beta)$ I shall call the "relative" angle between the two threads. Designate the relative angle between threads that makes θ and Φ equal by ψ. Divide (126) by (125):

$$\begin{aligned}\frac{\cos\Phi}{\cos\theta} &= \frac{[1+PQ\cos(A-B)]\cos(\alpha-\beta)-PQ\sin(A-B)\sin(\alpha-\beta)}{\sqrt{1+2PQ\cos(A-B)+P^2Q^2}}\\ &= \cos(\alpha-\beta-\psi),\end{aligned}$$

hence,

$$\cos\Phi = \cos\theta\cos(\alpha - \beta - \psi), \tag{129}$$

where ψ is determined by (127).

From (115):

$$\sin\theta = \pm\sqrt{\frac{(xy_1 - yx_1)(x\bar{y}_1 - y\bar{x}_1)}{(x\bar{x} + y\bar{y})(x_1\bar{x}_1 + y_1\bar{y}_1)}}. \tag{130}$$

The "angle of rotation" in transformation (106) can now be determined. The point (1, 0) is transformed into (a, c) hence,

$$\cos\theta = \sqrt{\frac{aa}{a\bar{a} + c\bar{c}}} = \sqrt{a\bar{a}}, \tag{131}$$

and

$$\sin\theta = \sqrt{c\bar{c}}, \tag{132}$$

(making a suitable convention regarding the direction of rotation).

The point (0, 1) is rotated into (b, d), hence

$$\cos\theta = \sqrt{d\bar{d}}, \tag{133}$$

$$\sin\theta = +\sqrt{b\bar{b}}. \tag{134}$$

Hence, taking into account (107) the transformation (106) can be written

$$\begin{aligned} x' &= x\cos\theta\, e^{i\alpha_1} - y\sin\theta\, e^{i(\alpha_1+\beta)}, \\ y' &= x\sin\theta\, e^{i\alpha_2} + y\cos\theta\, e^{i(\alpha_2+\beta)} \end{aligned} \tag{135}$$

and

$$ad - bc = e^{i(\alpha_1+\alpha_2+\beta)}.$$

If we search for invariant lines under (106) we shall discover that in general there are none. It is due to the fact that the Hermitian form $x\bar{x} + y\bar{y}$ is essentially positive and cannot vanish unless both x and y vanish and the determinant

$$\begin{vmatrix} a-1, & b \\ c\ , & d-1 \end{vmatrix}$$

does not in general vanish.

It is interesting to note that the geometry of the null plane under the generalized orthogonal group is identical with ordinary geometry under a certain subgroup of the metrical group in F_4 which is obtained if we take real and imaginary parts of x and y as coördinates of a point in F_4.

Some differential invariants under (106) are:

$$ds^2 = dx\,d\bar{x} + dy\,d\bar{y}, \tag{136}$$

$$\tan\theta = \sqrt{\frac{(dx\,\delta y - dy\,\delta x)(d\bar{x}\,\delta\bar{y} - d\bar{y}\,\delta\bar{x})}{(dx\,\delta\bar{x} + dy\,\delta\bar{y})(d\bar{x}\,\delta x + d\bar{y}\,\delta y)}}, \tag{137}$$

$$\cos\theta = \sqrt{\frac{(dx\,\delta\bar{x} + dy\,\delta\bar{y})(d\bar{x}\,\delta x + d\bar{y}\,\delta y)}{(dx\,d\bar{x} + dy\,d\bar{y})(\delta x\,\delta\bar{x} + \delta y\,\delta\bar{y})}}, \tag{138}$$

$$\cos\Phi = \frac{dx\,\delta\bar{x} + dy\,\delta\bar{y} + d\bar{x}\,\delta x + d\bar{y}\,\delta y}{2\sqrt{(dx\,d\bar{x} + dy\,d\bar{y})(\delta x\,\delta\bar{x} + \delta y\,\delta\bar{y})}} \tag{139}$$

the element of "length", the angle between curves, and the angle between "directions" on those curves; these can serve as a basis of definition of "curvature".

There are certain subgroups of (106) or (135) that should be mentioned. One of them is obtained if we let $\alpha_1 = \alpha_2 = \beta = 0$. It is an ordinary orthogonal group in a plane. The other subgroup is obtained if we let

$$\alpha_1 = 0,$$
$$\beta = \frac{\pi}{2},$$

and

$$\alpha_2 = -\frac{\pi}{2},$$

in which case equations (135) take the form:

$$\begin{aligned} x^1 &= x\cos\theta - iy\sin\theta, \\ y^1 &= -ix\sin\theta + y\cos\theta. \end{aligned} \tag{140}$$

Under (140) there are additional invariants, namely

$$x^2 - y^2, \quad xx_1 - yy_1, \quad xy_1 - yx_1 \tag{141}$$

with corresponding differential invariants

$$dx^2 - dy^2, \quad dx\,\delta x - dy\,\delta y, \quad dx\,\delta y - dy\,\delta x. \tag{142}$$

The following lines remain fixed under this subgroup:

$$y = x \quad \text{and} \quad y = -x. \tag{143}$$

The generalized orthogonal group is not the most general group of rotations in the null flat N_μ as I have assumed the metrical group in F_{2n} containing N_μ to be real. Take the following subgroup of rotations thru pure imaginary angle:

$$\begin{aligned} x_1^1 &= x_1 \cos i\theta - x_4 \sin i\theta, \\ x_2^1 &= x_2 \cos i\theta - x_3 \sin i\theta, \\ x_3^1 &= x_2 \sin i\theta + x_3 \cos i\theta, \\ x_4^1 &= x_1 \sin i\theta + x_4 \cos i\theta \end{aligned} \tag{144}$$

by which the null plane

$$\begin{aligned} x_3 &= ix_1, \\ x_4 &= ix_2 \end{aligned}$$

goes into itself and the transformations in it are:

$$\begin{aligned} x_1^1 &= x_1 \cosh\theta + x_2 \sinh\theta, \\ x_2^1 &= x_1 \sinh\theta + x_2 \cosh\theta. \end{aligned} \tag{145}$$

The geometry of the null plane under (145) is a kind of non-euclidean geometry that was discussed by Wilson and Lewis.*

* Wilson and Lewis, 'The space-time manifold of relativity', Proceedings of the American Academy of Arts and Sciences, vol. 48 (1913), pp. 387–507.

19. Consider a subflat F_n in F_{2n}:

$$x_{n+1} = k_1 x_1, \cdots, x_{2n} = k_n x_n. \tag{146}$$

The element of length in (146) will be

$$ds^2 = (1+k_1^2)\, x_1^2 + \cdots + (1+k_n^2)\, x_n^2. \tag{147}$$

Heretofore, I assumed

$$k_1 = \pm i, \cdots, k_n = \pm i,$$

i. e., that the matrix of the coefficients of the metrical fundamental form (147) is of rank 0.

If the matrix of the coefficients of the metrical fundamental form of a flat is of rank r the flat is said to be aeolotropic; I shall briefly designate it I_n^{n-r}

Let

$$k_1 = k_2 = \cdots = k_r = i$$

then the generalized element of length in I_n^{n-r} can be defined by:

$$\begin{aligned} ds^2 = k(dx_1\, d\bar{x}_1 + \cdots + dx_r\, d\bar{x}_r) \\ + l(dx_{r+1}^2 + \cdots + dx_n^2) \end{aligned} \tag{148}$$

where k and l are subject to arbitrary choice. The element of length so defined is invariant under rotations in F_{2n} that leave I_n^{n-r} invariant.

In I_2^1 we have:

$$ds^2 = k\, dx_1\, d\bar{x}_1 + l\, dx_2^2, \tag{149}$$

and also

$$k'\, dx_1\, \delta\bar{x}_1 + l'\, dx_2\, \delta x_2$$

as invariants, and others derivable from them. H. Beck assumed $k = l' = 0$ and $l = k' = 1$ and developed a special kind of geometry in the minimal plane.*

* H. Beck, 'Zur Geometrie in der Minimalebene', Sitzungsberichte der Berliner Math. Gesellschaft, vol. XII (1912).

20. In Chapters III and IV I have outlined a point of view from which the study of geometries in null and aeolotropic spaces can be pursued and incidentally I derived a few properties of the generalized orthogonal group. Certain problems like the problem of decomposition of a generalized rotation into component simple rotations in the invariant planes are left unsolved. The method I used in the case of the ordinary orthogonal group is not applicable here because of the absence of invariant lines. However, the considerations discussed in Chapter II can be extended to apply to a more general case. Still greater variety of problems relating to properties of curves and threads that were hardly touched in this paper remain for further study. The same refers to studies of relations between these geometries to those of higher linear monifolds of real points only. Thus, this work is merely an introduction into the subject which I intend to discuss more thoroughly in a subsequent paper.

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APPENDIX

In the following tables I shall use "c" and "s" in place of cos and sin and shall designate the angles (12), (13), (23), (14), (24), (34) successively by subscripts 1, 2, $\cdots$, 6, for instance cos (24) will be designated by c_5.

$n = 2$

$$a_1^1 = c_1, \quad a_1^2 = -s_1, \quad a_2^1 = s_1, \quad a_2^2 = c_1.$$

$n = 3$

$$a_1^1 = c_1 c_2, \quad a_1^2 = -s_1 c_2, \quad a_1^3 = -s_2,$$

$$a_2^1 = -c_1 s_2 s_3 + s_1 c_3, \quad a_2^2 = s_1 s_2 s_3 + c_1 c_3, \quad a_3^3 = -c_2 s_3,$$

$$a_3^1 = c_1 s_2 c_3 + s_1 s_3, \quad a_3^2 = -s_1 s_2 c_3 + c_1 s_3, \quad a_3^3 = c_2 c_3.$$

$n = 4$

$$a_1^1 = c_1 c_2 c_4, \quad a_1^2 = -s_1 c_2 c_4, \quad a_1^3 = -s_2 c_4, \quad a_1^4 = -s_4,$$

$$a_2^1 = -c_1 c_2 s_4 s_5 - c_1 s_2 s_3 c_5 + s_1 c_3 c_5,$$

$$a_2^2 = s_1 c_2 c_4 s_5 + s_1 s_2 s_3 c_5 + c_1 c_3 c_5,$$

$$a_2^3 = s_2 s_4 s_5 - c_2 s_3 c_5, \quad a_2^4 = -c_4 s_5,$$

$$a_3^1 = -c_1 c_2 s_4 c_5 s_6 + c_1 s_2 s_3 s_5 s_6 - s_1 c_3 s_5 s_6 + c_1 s_2 c_3 c_6 + s_1 s_3 c_6,$$

$$a_3^2 = s_1 c_2 s_4 c_5 s_6 - s_1 s_2 s_3 s_5 s_6 - c_1 c_3 s_5 s_6 - s_1 s_2 c_3 c_6 + c_1 s_3 c_6,$$

$$a_3^3 = s_2 s_4 c_5 s_6 + c_2 s_3 s_5 s_6 + c_2 c_3 c_6, \quad a_3^4 = -c_4 c_5 s_6,$$

$$a_4^1 = c_1 c_2 s_4 c_5 c_6 - c_1 s_2 s_3 s_5 c_6 + s_1 c_3 s_5 c_6 + c_1 s_2 c_3 s_6 + s_1 s_3 s_6,$$

$$a_4^2 = -s_1 c_2 s_4 c_5 c_6 + s_1 s_2 s_3 s_5 c_6 + c_1 c_3 s_5 c_6 - s_1 s_2 c_3 s_6 + c_1 s_3 s_6,$$

$$a_4^3 = -s_2 s_4 c_5 c_6 - c_2 s_3 s_5 c_6 + c_2 c_3 s_6, \quad a_4^4 = c_4 c_5 c_6.$$

VITA

Sergei Alexander Schelkunoff was born on the 27th of January in 1897 in the city of Samara, Russia. From 1905 till 1913 he attended "realnoye uchilishtse" in Orenbourg. Then he entered the Imperial School of Technology in Moscow; he was transferred to the University of Moscow in 1916. In 1917 he was enlisted in the Russian Imperial Army and sent to the military school in Odessa upon graduation from which he obtained a commission in a Siberian regiment. From 1918 on he was in far-eastern Siberia till he came to America in September 1921. From this date till June 1923 he attended the State College of Washington from which he received B. A. and M. A. degrees in Mathematics. From August 1923 till September 1926 he was a member of the research Staff of the Bell Telephone Laboratories, Inc. Since then he has been an instructor in mathematics at the State College of Washington. In 1923 he was elected a member of the Honorary Society of Phi Kappa Phi. During the academic years 1924–25 and 1925–26 he studied in the Graduate School of Columbia University.

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